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A CLASS OF SOLUTIONS TO THE GOSSIP PROBLEM, PART II

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We further develop the correspondence established in Part I of this paper to characterize optimal solutions to the gossip problem in which no one hears his own information. To reiterate, these are graphs on n vertices with $2n-4$ edges (n even), for which the edges can be linearly ordered to produce an increasing path from each vertex to every other, without having an increasing path from any vertex to itself. With two exceptions, we call these *NOHO-graphs*. In Part I, we associated each NOHO-graph with a quadruple consisting of two permutations and two binary sequences.

In Part II, we study the properties of the quadruples associated with NOHO-graphs. We characterize the quadruples that can arise. The main result is that any pair of sequences in such a quadruple uniquely determines the other pair. Using this, we count the realizable quadruples and several subclasses of them. There are $3^{(n-6)/2}$ realizable quadruples associated with NOHO-graphs on n vertices. The subclasses enumerated have various symmetry properties; their sizes are powers of 2 and 3. In Part II we also study the properties of a combining operator that we define for NOHO-graphs and their associated quadruples. In Part III, we will use these properties to enumerate the non-isomorphic NOHO-graphs.

"Don't advertise; tell it to a gossip."

Moses

1′. Introduction

In part I of this paper [1], we described the history of the gossip problem and numerous variations of it which have been studied. References to earlier work can be found there. The original gossip problem asked for the minimum number of telephone calls between pairs of n gossips so that each will learn everyone else's information; for $n \geqslant 4$ the answer is $2n-4$. We defined a new variation by adding the restriction that no gossip ever hears his own information.

The purpose of this section is two-fold. First, we summarize the results of Part I; second, we describe the results and methods used in the remainder of Part II. We will not repeat the particular phrasing we use to define standard graph theoretic concepts. The reader can find these definitions in Section 2 of Part I. The numbers appended to various results in this summary are the labels they received in Part I; if we need to refer to them subsequently, we will use the same labels.

Phrasing the gossip problem in graph terminology, we look for graphs whose edges are given a linear order such that there is an increasing path from each vertex x to every other vertex y. (We use $x \to y$ to mean '(an) increasing path from x to y'.) Such orderings on edge sets are called *pooling*. If there is no $x \to x$ for any vertex x, the graph (with edge ordering) *satisfies NOHO*. (Henceforth we substitute 'graph' for 'graph with edge ordering' when no confusion is possible.) Pooling graphs on n vertices can satisfy NOHO precisely when n is even and the smallest such graphs have $2n-4$ edges (Lemma 2.4).

NOHO prohibits triangles (Remark 2.2). We use $f(x)$ to denote the first neighbor of vertex x and $l(x)$ to denote its last neighbor. $F(G)$ is the collection of first-edges $(x, f(x))$; similarly $L(G)$ denotes the collection of last-edges. $F(G)$ and $L(G)$ each form a complete matching in an optimal graph, and they are disjoint (Lemma 2.3). The remaining set of 'middle' edges is denoted $M(G)$. $M(G)$, with $n-4$ edges, consists of four trees (Lemmas 3.1, 4.2). In drawing NOHO-graphs, we adopt the convention of drawing edges of $F(G)$ by dotted lines, edges of $L(G)$ by dashed lines, and edges of $M(G)$ by solid lines.

Among the optimal graphs, there are two 3-regular graphs on 8 vertices; all others have a 2-valent vertex (Theorem 3.4). We define a NOHO-graph to be an edge-ordered graph on n vertices and $2n-4$ edges that (a) is pooling, (b) satisfies NOHO, and (c) has a 2-valent vertex. For a NOHO-graph, the trees in $M(G)$ consist of two isolated vertices—the 2-valent vertices in G—and two caterpillars on $\frac{1}{2}n-1$ vertices each (Lemma 4.2). (A caterpillar is a tree with a path that contains or neighbors every vertex.)

Label the caterpillars of $M(G)$ as $\{C^1, C^2\}$. Each is a tree of increasing paths out of one vertex and a tree into another vertex. This enables us to define a *canonical numbering* of the vertices of a NOHO-graph, in which the vertices receive the labels $\{x^i_j: i=1,2;\ j=0,1,\ldots,m\}$. We will depend heavily on this canonical numbering hereafter. Assign x^i_0 to the 2-valent vertices. Label the vertices of C^i as x^i_j so that x^i_j is the jth to receive information from x^i_0. We also refer to x^i_0 as x^{3-i}_{m+1}, and we use $\bar{C}^i$ to denote the 'extended' caterpillar that consists of x^i_0, C^i, and x^i_{m+1}. The increasing path from x^i_0 to x^i_{m+1} in $\bar{C}^i$ is called its *distinguished path*.

The canonical numbering summarizes which increasing paths and adjacencies occur in the caterpillars. These properties (Remark 4.3) are listed below. (In addition to the notation $x \to y$, we use $x \sim y$ to mean 'x is adjacent to y' and use $x \nsim y$ for non-adjacency.)

(a) $\bar{C}^i$ contains $x^i_j \to x^i_k$ if and only if $j<k$ or $x^i_j \sim x^i_k$.

(b) x^i_k neighbors exactly one x^i_j such that $j<k$.

(c) If x^i_k neighbors any x^i_r with $r>k$, then it neighbors every x^i_j with $k<j<r$.

We associate with these caterpillars a binary *caterpillar sequence* $R(C)$. For a caterpillar whose vertices are indexed from 0 through $m+1$, the index set of the sequence is $\{1,\ldots,m\}$. $R_i=1$ if and only if the ith edge in the ordering belongs to the distinguished path. The *reverse caterpillar sequence* $R'(C)$ is the caterpillar sequence of the caterpillar with reverse edge-ordering; it equals the caterpillar sequence with indices reversed. We describe the placement of the edges in $M(G)$ by two binary sequences $S(G)$ and $T(G)$, where $S(G) = R(\bar{C}^1(G))$ and $T(G) = R'(\bar{C}^2(G))$. $S_j = 1$ if and only if x_j^1 lies on the distinguished path in $\bar{C}^1$, and $T_j = 1$ if and only if x_{m+2-j}^2 lies on the distinguished path in $\bar{C}^2$ (Remark 4.5). The first and $(m+1)$th elements of S and T are always 1, so we often drop them.

The placement of the first and last edges is described by two other sequences $P(G)$ and $Q(G)$. The first and last neighbors of a vertex in C^i must belong to $\bar{C}^i$, so we define $P_r = s$ if $f(x_r^1) = x_s^2$ and $Q_r = s$ if $l(x_r^1) = x_s^2$. P and Q are permutations; P on $\{1, \ldots, m+1\}$, Q on $\{0, \ldots, m\}$.

The main result of Part I is that the edges of any NOHO-graph can be described by such a quadruple of sequences $(P(G); Q(G); S(G); T(G))$. A quadruple that can arise as the defining quadruple of a NOHO-graph with respect to some ordering is called a *realizable quadruple* (Theorem 4.6). The remainder of Part I was devoted to showing constructively that NOHO-graphs are Hamiltonian (Theorem 5.2), bipartite (Lemma 5.3), and planar (Theorem 5.4). This holds also for the two 3-regular graphs on 8 vertices that we discarded.

Here in Part II, our object will be to characterize and count the realizable quadruples and several subclasses of them. A defining quadruple has two permutations and two binary sequences, but not all such quadruples are realizable. In Section 6, we develop necessary conditions for pairs of these integer sequences to belong to a realizable quadruple. Most of these conditions are technical ones relating the allowed positions of elements of various sequences. They rely on the properties of NOHO-graphs already developed and on two additional new concepts.

We define an operation called 'reflection', which in effect relabels the vertices of the graph so as to interchange the caterpillars. Also, we prove a result about 'reversions' in a permutation. We define a *reversion* to be a maximal consecutive subsequence of a permutation where the first element is the least. We call a reversion *simple* if it has the form $(i, i+r, i+r-1, \ldots, i+1)$. We prove that all the reversions of a permutation are simple if and only if the permutation has no three elements (not necessarily consecutive) of which the last is the largest. We prove that the latter condition always holds for $P(G)$.

Having derived enough necessary conditions, we show in Section 7 that any pair of sequences satisfying these necessary conditions uniquely determines the remaining pair. Furthermore, the resulting quadruple is realizable, so that the conditions are sufficient. Therefore, we need only count realizable pairs $(P; S)$, where P is the first-edge permutation and S is the sequence determining the first caterpillar. There are $\binom{p-1}{r-1}$ realizable first-edge permutations with r reversions (where $p = \frac{1}{2}(n-4)$), and 2^{r-1} realizable binary sequences for each of those. So, the binomial theorem gives us 3^{p-1} realizable quadruples.

In Section 8 we consider symmetric NOHO-graphs. When the operation of reflection does not change the quadruple, the graph is symmetric. Otherwise, two quadruples arise by reflection from a single graph. We characterize the first-edge and the last-edge permutations that can be associated with a symmetric NOHO-graph and count the number of 'symmetric quadruples' determined by them. We obtain altogether $3^{\lfloor p/2 \rfloor}$ symmetric NOHO-graphs.

Section 9 treats 'concatenation'. Concatenation of two NOHO-graphs deletes one vertex of degree two from each, identifies two pairs of vertices, and merges the edge-orderings in a natural way. The resulting graph has four fewer vertices than the union of the two original graphs. This explains why the quantity $p = \frac{1}{2}(n-4)$ appears so frequently; it is additive under concatenation. In Section 9 we show that the concatenation of two NOHO-graphs is also a NOHO-graph. To find the quadruple describing a concatenated graph, it suffices to generate $(S; T)$ by concatenating the S sequences and the T sequences from the component graphs and then generate $(P; Q)$ from $(S; T)$. Also, a realizable $(S'; T')$ can be 'split off' from the front of any realizable quadruple that 'contains' it, leaving a realizable quadruple on the remaining part of $(S; T)$.

In Section 10 we examine irreducible NOHO-graphs—those which cannot be formed by concatenation. We show there is unique decomposition of any NOHO-graph as a concatenation of irreducible ones. This follows because the 'least common refinement' (in terms of compositions of integers) of two such decompositions is also a decomposition. Using induction we show that there are $\binom{p-1}{k-1}2^{p-k}$ realizable quadruples formed from k irreducible parts. We have no closed form for the number of symmetric realizable quadruples formed from k irreducible parts. However, in the special case of irreducible symmetric solutions ($k = 1$), the sum can be computed. The number of these is $2^{\lfloor p/2 \rfloor}$.

Finally, in Section 11 we return to conditions for realizable quadruples. In Sections 6 and 7 we did not give the necessary and sufficient conditions for realizability of the pairs $(S; T)$ because they are mostly easily expressed for irreducible quadruples. The characterization is surprisingly simple, and it explains more directly why the numbers of realizable quadruples and symmetric quadruples are powers of 3. It also enables us to enumerate the 'reversible' quadruples (those where reversing the edge-ordering on the graph gives rise to the same defining quadruple). In Part III, this characterization will be our primary tool for determining which realizable quadruples can arise from isomorphic NOHO-graphs. That will lead, at last, to an enumeration of the non-isomorphic NOHO-graphs.

6. Necessary conditions for realizability

We begin our examination of the properties of realizable quadruples by deriving necessary conditions for pairs of sequences in $(P; Q; S; T)$ to belong to realizable quadruples. The first lemma was used in Section 5, but it will be used often in the subsequent sections and so bears explicit treatment here.

Notation. To avoid subscripted subscripts, we will sometimes write $R(i)$ instead of R_i for the ith element of a sequence.

Lemma 6.1. *If G is a NOHO-graph, then $P(G)$ and $Q(G)$ satisfy*

(a) $j=k \Rightarrow P_j > Q_k$.
(b) $P_j = Q_k \Rightarrow j > k$.

Proof. Due to the canonical numbering, $r<s$ implies there exists $x_r^i \to x_s^i$ completely within $\bar{C}^i$. This path uses no first or last edges. Violating (a) creates $x_j^1 \to x_{P(j)}^2 \to x_{Q(j)}^2 \to x_j^1$, while violating (b) creates $x_{P(j)}^2 \to x_j^1 \to x_k^1 \to x_{Q(k)}^2 = x_{P(j)}^2$. $\square$

Corollary 6.2. *In a realizable quadruple, $Q_0 = P_2$. Also, $Q_j = 0$ if and only if $P_j = 2$.*

Proof. $P_1 = m+1$, so $1 \leq P_2 \leq m$, and P_2 appears in Q. By Lemma 6.1(b), it must be Q_1 or Q_0. If the former, then $f(x_2^1) = l(x_1^1)$, and $(x_1^1, x_2^2, f(x_2^1))$ is a triangle. Similarly, if $P_k = 2$ but $Q_k \neq 0$, Lemma 6.1(a) requires $Q_k = 1$. Now $f(x_2^2) = l(x_1^2)$, and $(x_1^2, x_2^2, f(x_2^2))$ is a triangle. $\square$

These permutations are not independent of the caterpillar sequences. In fact, every increasing pair in $P(G)$ forces two particular edges to appear in the caterpillars, while an increasing pair in $Q(G)$ forces at least one of two edges to appear. This asymmetry arises because we have numbered the caterpillars as out-trees.

Lemma 6.3. *For a NOHO-graph G, $P(G)$ and $Q(G)$ satisfy*

(a) *If $P_i < P_j$ with $i<j$, then (x_i^1, x_j^1) and $(x_{P(i)}^2, x_{P(j)}^2)$ are edges.*
(b) *If $Q_i < Q_j$ with $i<j$, then at least one of (x_i^1, x_j^1) and $(x_{Q(i)}^2, x_{Q(j)}^2)$ is an edge.*

Proof. Consider any increasing pair in P. Let $r = P_i$ and $s = P_j$, where $i<j$ and $r<s$. First suppose $x_i^1 \not\sim x_j^1$, as in Fig. 6.1(a). Since $i<j$ and $x_i^1 \not\sim x_j^1$, information from x_j^1 can reach x_i^1 only via $\bar{C}^2$. To cross over and return, the path must use the first call of x_j^1 and the last call of x_i^1. Let $t = Q_i$, so $x_t^2 = l(x_i^1)$. Lemma 6.1(a) implies $t = Q_i < P_i = r$. With $t<r<s$, $x_s^2 \to x_t^2$ exists within $\bar{C}^2$ only if $x_s^2 \sim x_t^2$. By the canonical numbering (Remark 4.3(c)), this requires that also $x_r^2 \sim x_t^2$, making (x_i^1, x_r^2, x_t^2) a triangle.

If $x_s^2 \not\sim x_r^2$, we switch the roles of C^1 and C^2 in the argument above. By similar reasoning, completing $x_s^2 \to x_r^2$ forces the triangle $(x_r^2, x_j^1, l(x_r^2))$

Finally, suppose $Q_i < Q_j$ with $i<j$ but $x_i^1 \not\sim x_j^1$, as in Fig. 6.1(b). Having $x_j^1 \to x_i^1$ again requires $x_{P(j)}^2 \to x_{Q(i)}^2$ in $\bar{C}^2$. Lemma 6.1(a) implies $Q_i < Q_j < P_j$, so completing that path requires $x_{P(j)}^2 \sim x_{Q(i)}^2$. With the canonical numbering (Remark 4.3(c)), this implies also $x_{Q(i)}^2 \sim x_{Q(j)}^2$, as desired. $\square$

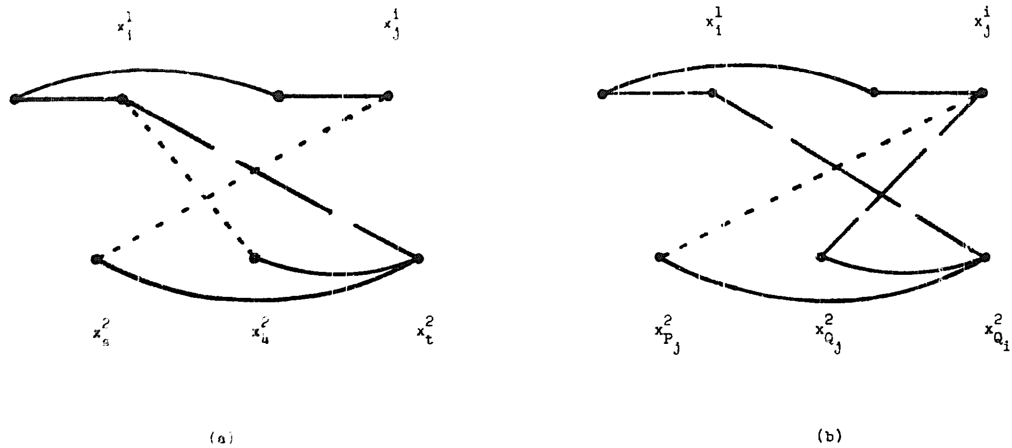


Fig. 6.1. Increasing pairs in (a) P and (b) Q

To completely characterize the permutations that arise as $P(G)$, we need a result about permutations. We define a *reversion* in a permutation to be a maximal consecutive subsequence of the permutation where the first element is the least. The reversions of a permutation partition it into segments. They correspond to the cycles of another permutation in one well-known canonical representation of permutations. In that representation, cycles are written as consecutive subsequences in descending order by least elements, and within each cycle the least element is written first. We call a reversion *simple* if it consists of a single element or has the form $(r, s, s-1, \ldots, r+1)$ with $s-r+1$ elements.

Lemma 6.4. *Suppose P is a permutation of a consecutive sequence of numbers. Then every reversion of P is simple if and only if P has no subsequence of length 3 whose last element is its largest.*

Proof. The first elements of the reversions in any permutation form a decreasing subsequence, else the reversions would not be maximal. If also they are all simple, then any increasing subsequence must lie entirely within a single reversion. The form of a simple reversion prohibits two increasing pairs with the same second element.

Conversely, assume there is no bad triple, and consider a reversion $(r, s, \ldots)$. Dropping r from the reversion must leave a strictly decreasing subsequence, since any increasing pair would have made a bad triple with r. Suppose there is some element t with $r<t<s$ that does not appear in the reversion. If t appears before r, then (t, r, s) is a bad triple. If t appears after this reversion, let r' be the first element of the reversion containing t. Then (r, r', t) forms a bad triple, since the first elements of reversions from a decreasing subsequence. $\square$

Lemma 6.5. *If G is a NOHO-graph, then every reversion of $P(G)$ is simple.*

Proof. We need only show P has no 3-element subsequence P_i, P_j, P_k $(i<j<k)$ whose last element is largest. If so, then $x_i^1 \sim x_k^1$ and $x_j^1 \sim x_k^1$, by Lemma 6.3. By the canonical numbering (Remark 4.3(c)), $x_i^1 \sim x_k^1$ forces $x_i^1 \sim x_j^1$, which creates a triangle. □

Corollary 6.6. *If the reversions of a permutation are all simple, then the permutation is uniquely determined by choosing the subset of the non-initial indices where reversions will begin. In particular, there are 2^{m-2} permutations realizable as $P(G)$ for some NOHO-graph G.*

Proof. The first reversion always begins at the first index. Iteratively, each simple reversion is determined by its length. In $P(G)$, $P_1 = m+1$ and $P_{m+1} = 1$, so reversions always begin in positions 2 and $m+1$, also. □

This fact is not superfluous, because the necessary condition of Lemma 6.5 will suffice for P to be realizable as $P(G)$. Also, adding Lemma 6.1 will suffice for the pair $(P; Q)$ to be realizable. Next, we derive a necessary condition for the pair $(P; S)$.

Lemma 6.7. *Consider $P(G)$ and $S(G)$ for some NOHO-graph G. Suppose that P_{j+1} does not begin a reversion or that $j = m$. Then $S_j = 1$ if and only if P_j begins a reversion. Equivalently, when P_{j+1} does not begin a reversion, x_j^1 lies on the path V from x_0^1 to x_{m+1}^1 if and only if P_j begins a reversion in P.*

Proof. The equivalence follows from the canonical numbering and the definition of $S(G)$, as noted in Remark 4.5. Now, if P_{j+1} does not begin a reversion, we can assume P_j lies in a reversion $(P_i, \ldots, P_k)$ of length at least two. Since by Lemma 6.5 it is a simple reversion, every element after P_i forms an increasing pair with P_i. By Lemma 6.3(a), every vertex from x_{i+1}^1 through x_k^1 is joined to x_i^1. Since x_i^1 is the *first* of these vertices to receive information from x_0^1, it must lie on V. Also, since the vertices are numbered 'out from x_0^1', there would be no $x_k^1 \rightarrow x_{m+1}^1$ in $\bar{C}^1$ if one of $\{x_{i+1}^1, \ldots, x_{k-1}^1\}$ were on V. When $j = m$, x_{j+1}^1 is automatically on V. The same arguments then show x_m^1 is also on the path if and only if P_m begins a reversion, i.e. $P_m = 2$. □

Most of our enumerative remarks will come later, but here we digress for an immediate consequence of this lemma.

Corollary 6.8. *For each P whose r reversions are all simple, the number of sequences S satisfying Lemma 6.7 is 2^{r-1}.*

Proof. An element of S is unrestricted if and only if its position corresponds to the last element of a reversion in P (other than P_{m+1}). □

To obtain necessary conditions for other pairs from these four sequences, we need an additional operation. It corresponds to interchanging the roles of C^1 and C^2 and looking at the graph 'upside down'. Denote by $(P^*; Q^*; S^*; T^*)$ the *reflection* (or reflected quadruple) of $(P; Q; S; T)$, defined by setting $P_i^* = j$ if $P_j = i$, $Q_i^* = j$ if $Q_j = i$, $S_i^* = T_{m+2-i}$, and $T_i^* = S_{m+2-1}$. A little 'reflection' gives the following remark.

Remark 6.9. *The reflection of a realizable quadruple is realizable by the same NOHO-graph.*

Remark 6.10. *If G is a NOHO-graph, then $(P(G); T(G))$ must be such that $(P^*(G); S^*(G))$ satisfies Lemmas* 6.5 *and* 6.7.

The *reverse graph* $K(G)$ of a NOHO-graph G is also a NOHO-graph, as remarked in Part I. This is the graph with the same vertices and edges as G, but with $(x, y) < (u, v)$ in $K(G)$ if and only if $(x, y) > (u, v)$ in G. To obtain the canonical numbering for $K(G)$, the vertices need to be renumbered within C^1 and C^2. The renumbering is more complicated than simply reversing the order of the vertices within the caterpillar. Even so, we remarked earlier that reversing the edge order of a caterpillar simply reverses its caterpillar sequence. In other words (see Fig. 6.2),

G K(G)

Fig. 6.2. $PQST(G) = (7654231; 6453201; 01010; 10111)$,
$PQST(K(G)) = 7645231; 6543201; 01010; 11101)$.

Remark 6.11. *For a NOHO-graph G, $S(K(G)) = T^*(G)$ and $T(K(G)) = S^*(G)$.*

$P(K(G))$ and $Q(K(G))$ are not as easy to read off. In fact, a single P or Q can lead to more than one P or Q in the reverse graph, depending on the other sequences. Nevertheless, reversing can expand Lemmas 6.1, 6.5, 6.7, and Remark 6.10 to necessary conditions for the remaining pairs of sequences, except for $(S; T)$. They are not needed, and we omit them here. We postpone discussing the condition for $(S; T)$ until Section 11, when the meaning of the condition will be much clearer. Until then, our conditions on the pair $(P; S)$ will 'suffice'.

7. The number of realizable quadruples

Besides showing the sufficiency of the previous conditions, we will show that any pair from $(P; Q; S; T)$ that satisfies the relevant conditions is realized by a *unique* NOHO-graph. Thus, enumerating the realizable pairs $(P; S)$ will give us the number of realizable quadruples. The first step toward this is to show how to generate one of $(P(G); Q(G); S(G))$ when given the other two. We can apply the same algorithm to the reflected sequences $(P^*; Q^*; S^*)$ to obtain the same result for $(P(G); Q(G); T(G))$. This will leave only generating $(P(G); Q(G))$ from $(S(G); T(G))$.

Notation. $S(G)$ is a binary sequence. On its index set we define a function b that points to the previous 1 in the sequence. For $2 \leqslant i \leqslant m$, let $b(i)$ be the greatest positive integer such that $j < i$ and $S_j = 1$. Note this is well-defined, since S_1 always equals 1. The basic structural result relating these sequences is

Lemma 7.1. *For a NOHO-graph G, $(P(G); Q(G); S(G))$ are related by*

(a) $S_i = 1$ *if and only if* $P_{i+1} = Q_{b(i)}$.

(b) $S_i = 0$ *if and only if* $P_{i+1} = Q_i$.

Proof. In one direction the lemma is easy. Suppose P_{i+1} equals $Q_{b(i)}$ or Q_i. Since $S_{b(i)} = 1$, Remark 4.5 implies that $b(i)$ is the highest index below i of a vertex on the distinguished path in C^1. By the canonical numbering, $x^1_{b(i)}$ is the unique vertex of C^1 that precedes and neighbors x^1_i. Again by Remark 4.5, $S_i = 0$ if and only if x^1_i is not on the distinguished path, in which case $x^1_{b(i)} \sim x^1_{i+1}$. On the other hand, $S_i = 1$ if and only if x^1_i is on the path, so instead $x^1_i \sim x^1_{i+1}$. Therefore, if P_{i+1} is $Q_{b(i)}$ or Q_i, then choosing S_i the wrong way creates a triangle using x^1_{i+1}, $x^2_{P(i+1)}$, and $x^1_{b(i)}$ or x^1_i.

The converse follows by induction on i. We need only show that P_{i+1} must always be $Q_{b(i)}$ or Q_i. If not, let k be the least $i \geqslant 2$ where this fails in some realizable quadruple. By Lemma 6.1 we know that $P_{k+1} = Q_j$ for some $j \leqslant k$. Since the assertion fails, we have $j < k$ and $j \neq b(k)$. Now if $S_j = 0$ we have $P_{j+1} = Q_j$, contradicting $j < k$. So suppose $S_j = 1$; we will again show that Q_j must show up in P earlier than P_{k+1}.

Since $S_j = 1$ and $j < k$, the definition of b requires $j \leqslant b(k)$. We have assumed $j \neq b(k)$. Let t be the least integer greater than j such that $S_t = 1$, so $j = b(t)$. Since b is monotonic and $j < b(k)$, we have $t < k$, so the assertion holds for t. Now $S_t = 1$ yields $P_{t+1} = Q_{b(t)} = Q_j$, again a contradiction. □

The following corollary is a graphical version of the preceding lemma and aids greatly in drawing the NOHO-graphs that realize particular quadruples. Both results are illustrated by Fig. 7.1.

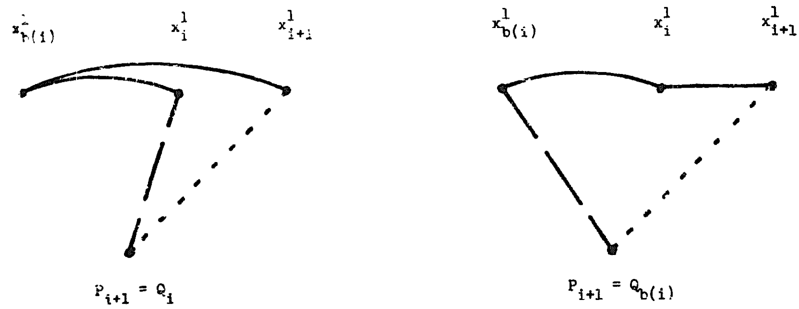


Fig. 7.1. $S_i = 0$ and $S_i = 1$.

Corollary 7.2. *The first and last neighbors of any point in the caterpillar C^i are joined by a path of two edges in the opposite (extended) caterpillar $\bar{C}^j$.*

Proof. Look first at x_r^2, $1 \leq r \leq m$. We have $P_{k+1} = Q_{b(k)} = r$ with $S_k = 1$ or $P_{k+1} = Q_k = r$ with $S_k = 0$. In either case, the edges $(x_k^1, x_{b(k)}^1)$ and $(x_{k+1}^1, x_{b(k+1)}^1)$ form the desired path. For x_s^1, the result follows by reflection. □

Note also that the first and last neighbors of x_0^i are joined by (at least) two paths of length two.

Now we proceed to the main results.

Theorem 7.3. *Any pair of sequences from a realizable quadruple $(P; Q; S; T)$ uniquely determines the remaining pair.*

Proof. We provide algorithms to generate the unique missing sequences that will satisfy all the necessary conditions.

Suppose the two known sequences lie in $(P; Q; S)$. Given $(P; Q)$ satisfying Lemmas 6.1 and 6.5, we generate the only S that satisfies Lemma 7.1. Initialize $k = 1$. Then for $i = 2, 3, \ldots, m$, if $P_{i+1} = Q_k$, set $S_i = 1$ and reset $k = i$. If $P_{i+1} = Q_i$, set $S_i = 0$ and leave k unchanged. As we proceed in P, the only previous elements of Q that have not been encountered are Q_k and Q_i, so these are the only choices available. The choice made for S_i is the only one consistent with Lemma 7.1.

The resulting $(P; S)$ satisfies Lemma 6.7. If not, let k be the least index where it fails. This means P_{k+1} does not begin a reversion. Also, $S_k = 1$ with P_k not beginning the reversion, or $S_k = 0$ with P_k beginning the reversion. If the latter, then $P_k < Q_k$. If the former, then the reversion containing P_k, P_{k+1} starts at $P_{b(k)}$, since k was the least index where Lemma 6.7 failed. But then $Q_{b(k)} = P_{k+1}$, so $P_{b(k)} < Q_{b(k)}$.

Next, given (P, S) satisfying Lemmas 6.5 and 6.7, we generate the only Q that will satisfy Lemma 7.1. Set $Q_i = 0$ if $P_i = 2$, and $Q_0 = P_2$ (Corollary 6.2). For all other i, if $S_i = 0$ set $Q_i = P_{i+1}$, while if $S_i = 1$ set $Q_{b(i)} = P_{i+1}$. This is well-defined. The indices skipped by the first option are those with $S_i = 1$, so the subsequence consisting of those elements merely shifts within itself as it passes from P to Q. $P_2 = 0$ makes room for the shift (i.e., 1 is the least $b(i)$). Having $Q_i = 0$ where $P_i = 2$ fills the hole at the other end, since that is the index of the last 1 in S. Again, any other choice for Q will violate Lemma 7.1.

The resulting $(P; Q)$ satisfies Lemma 6.1. That $P_i = Q_j$ implies $i > j$ is obvious by construction. On the other hand, suppose some $P_j < Q_j$. The algorithm sets $Q_j = P_{i+1}$ for some $i > j$, so by Lemma 6.5 P_j must begin a reversion containing P_{i+1}. By Lemma 6.7 $S_j = 1$, so in the construction Q_j is set the next time a 1 is encountered in S, i.e. $j = b(i)$ and $S_i = 1$. But then Lemma 6.7 implies P_{i+1} must be in a later reversion than P_j.

For the remaining cases, we give less detail. To generate P from $(Q; S)$, set $P_1 = m+1$, $P_{m+1} = 1$, and $P_2 = Q_0$. For all other i, if $S_{i-1} = 0$ set $P_i = Q_{i-1}$, while if $S_{i-1} = 1$ set $P_i = Q_{b(i-1)}$. This is well-defined since the only previous elements of Q not placed at the ith stage are Q_{i-1} and $Q_{b(i-1)}$. The resulting $(P; Q)$ clearly satisfies Lemma 6.1 and can be shown to satisfy Lemma 6.5. By the construction, they also satisfy Lemma 7.1.

To generate T from $(P; Q; S)$, reflect to get $(P^*; Q^*)$ and use the first algorithm above to get S^*. Then $T_i = S^*_{m+2-i}$.

To generate the unknown sequences knowing T and one of $(P; Q)$, reflect them and apply the above algorithms for S and one of $(P; Q)$. This generates T^* and the unknown element of $(P^*; Q^*)$, and reflecting again gives the desired quadruple.

This leaves the case of generating $(P; Q)$ from a realizable $(S; T)$. We will generate the first edges and last edges using the four-cycles described in Corollary 7.2. The order in which we generate the edges gives the paths discussed in Section 5, where they were used to construct a Hamiltonian cycle and a planar representation for the NOHO-graph.

Begin with $f(x^2_{m+1}) = x^1_1$. Build a path that alternates between vertices of C^1 and C^2, traveling to C^2 on last edges and to C^1 on first edges. Having reached x^1_i by $x^1_i = f(x^2_{j+1})$, set $l(x^1_i) = x^2_j$ if $T_{m+2-j} = 0$. Otherwise set $l(x^1_i) = x^2_k$ if $T_{m+2-j} = 1$ and $m+2-k$ is the position of the next 1 in T. Similarly, having reached x^2_j by $x^2_j = l(x^1_i)$, set $f(x^2_j) = x^1_{i+1}$ if $S_i = 0$. Otherwise set $f(x^2_j) = x^1_k$ if $S_{k-1} = 1$ and $i = b^2(k)$. An example of the path generated is shown in Fig. 7.2. To generate the remaining

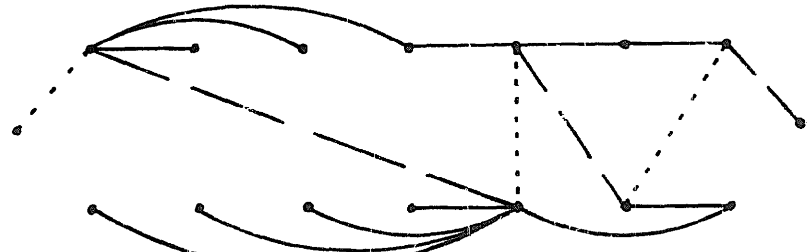


Fig. 7.2. The path from x^2_{m+1} generated by $(S; T) = (001111; 100010)$.

first and last edges, perform the same procedure beginning with $l(x_0^1) = x_r^2$, where T_{m+2-r} is the first 1 in T.

It was shown in Section 5 that each of these two paths reaches x_0^2 and that their union forms a Hamiltonian circuit containing all the first edges and last edges. Here we have constructed them using only edges guaranteed (by Lemma 7.1 and Corollary 7.2) to exist in any realization of $(S; T)$. Thus, even though we have not stated the conditions that $(S; T)$ must satisfy, we have generated from them a unique realizable $(P; Q)$. □

We have shown uniqueness, now we show sufficiency. No matter what pair we began with, as long as they satisfied the appropriate necessary conditions, we have generated a quadruple that satisfies all the necessary conditions derived in Section 6. The quadruple prescribes the edges and edge-ordering of a graph G. The necessary conditions together imply that increasing paths exist between all ordered pairs of vertices and that NOHO is satisfied, as seen next.

Theorem 7.4. *A quadruple of sequences $(P; Q; S; T)$ is realizable by a NOHO-graph if it satisfies the necessary conditions of Lemmas* 6.1, 6.5, 6.7, *and Remark* 6.10.

Proof. $(P; Q; S; T)$ determines the placement of $2n-4$ edges in a labelled graph on n vertices, where the remainder of the edge-ordering is required to conform to the canonical numbering. Therefore, $x_j^i \to x_k^i$ exists when $j<k$. Next we show $x_j^1 \to x_k^2$ exists. Let $s = P_j$, and define r by $Q_r = k$. If $s \leqslant k$ or $r \leqslant j$, we are finished. If both of these possibilities fail, we claim $x_j^1 \sim x_r^1$ or $x_k^2 \sim x_s^2$, so that (x_j^1, x_r^1, x_k^2) or (x_j^1, x_s^2, x_k^2) is the desired path. Define t by $P_t = k$. If $t<j$, then (P_t, P_j) form an increasing subsequence of P, and P_t begins a reversion that includes P_j (Lemma 6.5). To satisfy Lemma 6.7, all the vertices named in the reversion, including $x_s^2 = f(x_j^1)$, must be joined to x_k^2 in C^2, as desired. If instead $t>j$, we can apply Lemma 7.1, which followed from these necessary conditions. Since $r<j<t$, we have $t \neq r+1$ but $P_t = Q_r$, so we must have $r = b(t-1)$ and $S_{t-1} = 1$. So, $x_r^1 \sim x_{t-1}^1$. By the canonical numbering, this also requires $x_r^1 \sim x_j^1$, as claimed. These cases are illustrated in Fig. 7.3.

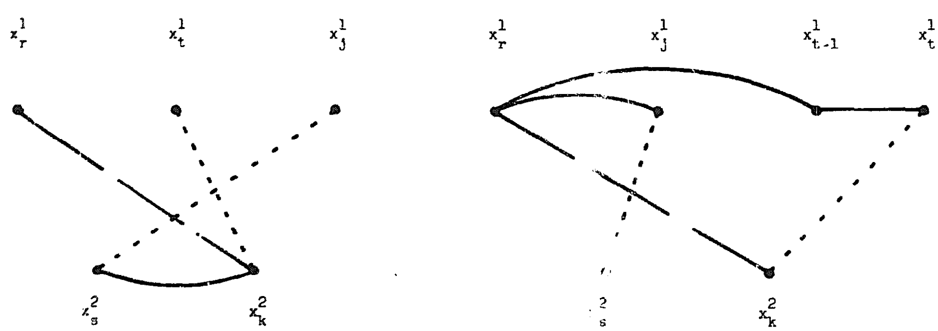


Fig. 7.3. $x_j^1 \to x_k^2$ when $t<j$ or $t>j$.

We also have $x_j^1 \to x_r^1$, even if $r<j$. Again let $s=P_j$ and $k=Q_r$. If $x_j^1 \sim x_r^1$ or $x_s^2 \sim x_k^2$ or if $s \leq k$, then we are done. In considering $x_j^1 \to x_k^2$ above, we showed that if $r<j$ and $s>k$, then we must have $x_j^1 \sim x_r^1$ or $x_s^2 \sim x_k^2$.

That all increasing paths from x_j^2 exist follows from reflection.

As constructed, G trivially satisfies NOHO. $v \to v$ cannot occur using the edges in a single caterpillar, so it must cross to $f(v)$ and return from $l(v)$. Suppose $f(v)=x_j^i$ and $l(v)=x_k^i$. Completing the path requires $x_j^i \sim x_k^i$ or $j \leq k$. The former never occurs because we've constructed a graph with no triangles, and the latter never occurs because $(P; Q)$ satisfies Lemma 6.1. So, the graph determined by $(P; Q; S; T)$ is a NOHO-graph. □

Theorem 7.5. *The number of quadruples realizable by NOHO-graphs on $2m+2$ vertices is 3^{m-2}, where $m \geq 2$.*

Proof. Using the methods in Theorem 7.3, a pair $(P; S)$ satisfying Lemmas 6.1, 6.5, 6.7 generates a unique $(Q; T)$ satisfying Remark 6.10. The resulting quadruple is realizable, so we need only count the number of pairs $(P; S)$ satisfying the first three conditions. As noted in Corollary 6.8, a realizable P can be paired with 2^{r-1} possible S to satisfy Lemma 6.7, where r is the number of reversions from P_2 through P_m. By Corollary 6.6, there are $\binom{m-2}{r-1}$ such realizable P. Using the binomial theorem, the total number of realizable quadruples is $\sum \binom{m-2}{r-1} 2^{r-1} = 3^{m-2}$. □

Fig 7.4. exhibits the quadruples and associated graphs for $n=6$ and $n=8$. We omit the leading and trailing 1's for S and T, unless otherwise indicated.

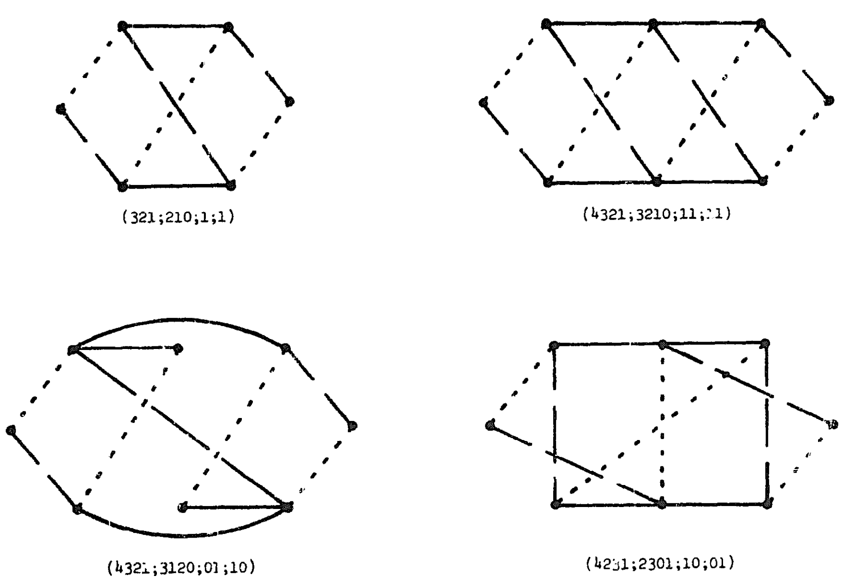


Fig. 7.4. Small NOHO-graphs.

G has 180° rotational symmetry when drawn as in Fig. 7.3 if and only if $(P^*; Q^*; S^*; T^*) = (P; Q; S; T)$. This occurs for all the graphs in Fig. 7.3. If this symmetry does not hold, then G is counted (at least) twice when the quadruples are enumerated. In the next section we enumerate the symmetric solutions.

8. Symmetric NOHO-graphs

In this section we count the symmetric NOHO-graphs. We define a *symmetric quadruple* as a realizable quadruple for which $(P^*; Q^*; S^*; T^*) = (P; Q; S; T)$. A *symmetric NOHO-graph* is a NOHO-graph for which the vertex permutation interchanging x_k^i and x_k^j for all k leaves the graph unchanged. This differs from the usual definition of graph symmetry because we are interested in a particular vertex permutation. In a sense, we are including the edge-ordering in the concept of symmetry. As noted above,

Remark 8.1. *G is a symmetric NOHO-graph if and only if $(P(G); Q(G); S(G); T(G))$ is a symmetric quadruple.*

The following remark applies to all $P(G)$, and is useful in determining the number of symmetric ones.

Remark 8.2. *In a realizable P, $P_i = j$ implies $j \leq m+3-i$, with equality if and only if P_i does not begin a reversion.*

Proof. Since all the reversions are simple, at most one element less than P_i can precede P_i, and none such will precede it unless P_i does not begin a reversion. The remaining elements less than P_i must follow P_i, and there must be room for them, so $m+1-i \geq j-2$. □

Lemma 8.3. *The number of symmetric realizable $P(G)$ is $2^{\lfloor (m-1)/2 \rfloor}$.*

Proof. Symmetry requires $P_j = i$ if $P_i = j$, so P corresponds to a matching of the positions $(1, \ldots, m+1)$. 1 always matches to $m+1$. Some positions may be matched to themselves, i.e. $P_i = i$, but we will see this can only happen twice in a given P.

We construct P match by match from position m down to position $\lceil \frac{1}{2}(m+3) \rceil$. At each step we claim there are two choices. Since all the reversions are simple, $P_m \in \{2, 3\}$, and in fact $P_{m-k} \in \{2, 3, \ldots, m-k+3\}$. However, k of these have already been matched with higher positions on previous steps. This leaves two choices for P_{m-k}: the newly-available $m-k+3$, and one lower value previously unchosen. On reaching $P_{\lceil (m+3)/2 \rceil}$, the choices are $\lfloor \frac{1}{2}(m+3) \rfloor$ and one lower value. If m is odd, we choose between matching them to each other or to themselves. If m

is even, match $\lceil\frac{1}{2}(m+3)\rceil$ to one of them and match the remaining one to itself. Now we have made $m-\lceil\frac{1}{2}(m+3)\rceil+1=\lfloor\frac{1}{2}(m-1)\rfloor$ choices and completed the matching. Every P so constructed satisfies Lemma 6.5, and these are all the symmetric P that do so. Furthermore, they are all realizable. □

Notation. Examining the construction in the proof above, we can define a binary sequence $B(P)$, indexed from $\lceil\frac{1}{2}(m+3)\rceil$ to m, where $B_{m-k}=0$ if $P_{m-k}=m-k+3$ and $B_{m-k}=1$ if $P_{m-k}<m-k+3$. Now we can count the quadruples associated with each symmetric P.

Lemma 8.4. *Suppose P is realizable as $P(G)$ for a symmetric NOHO-graph G. Then the number of symmetric quadruples realizing P is 2^q, where q is the number of 1's in $B(P)$.*

Proof. We consider how many ways symmetric Q can be constructed so that $(P;Q)$ satisfies Lemma 6.1. We claim that each way determines a unique symmetric quadruple. By Theorems 7.4 and 7.5 each determines a unique realizable quadruple. Using the algorithms in Theorem 7.4, we generate S and T. Since P and Q are symmetric, we get the same result if we reflect and apply the algorithms again. In other words, $S=S^*$ and $T=T^*$. Therefore, each generated quadruple is symmetric, and all the symmetric ones appear.

To construct Q, we first construct the forced part of S. Recalling Lemma 6.7, an element of S is unrestricted if and only if it precedes a first element of a reversion in P other than P_{m+1}. So, covering the index range $\lceil\frac{1}{2}(m+1)\rceil$ to m, there are 2^q ways to write down this portion of a realizable $(P;S)$. Using the algorithm in Corollary 7.2, we can write down what the corresponding segment of Q must be.

We claim there is a unique way to complete Q to be both symmetric and realizable. $Q_{\lceil(m+1)/2\rceil},\dots,Q_m$ have already been defined. Now, for $k\geq\frac{1}{2}(m+1)$, set $Q_j=k$ if $Q_k=j$. This completes Q, except possibly for two positions. First we show this operation is well-defined; then we fill the remaining positions.

Only if $Q_k\geq\frac{1}{2}(m+1)$ can we have a conflict. We always have $k+Q_k\leq m+2$, since $Q_k<P_k$ and $k+P_k\leq m+3$. This means we can have both $Q_k\geq\frac{1}{2}(m+1)$ and $k\geq\frac{1}{2}(m+1)$ only if (1) $Q_{\lceil(m+1)/2\rceil}=\lceil\frac{1}{2}(m+1)\rceil$ or (2) m is odd and $Q_{(m+1)/2}=\frac{1}{2}(m+3)$ or $Q_{(m+3)/2}=\frac{1}{2}(m+1)$. The first possibility is automatically symmetric. We need only show that either part of the latter case forces the other, which again is symmetric.

First suppose $Q_{(m+1)/2}=\frac{1}{2}(m+3)$. Due to $k+P_k\leq m+3$, $P_k>Q_k$, symmetry of P, and $P_i=Q_j$ implying $i>j$, we have $(P_{(m+1)/2},P_{(m+3)/2},P_{(m+5)/2})=(\frac{1}{2}(m+5),\frac{1}{2}(m+3),\frac{1}{2}(m+1))$. Since $k+P_k=m+3$ in these positions, $B_{(m+3)/2}=B_{(m+5)/2}=0$. Neither $P_{(m+3)/2}$ nor $P_{(m+5)/2}$ begins a reversion. Thus $S_{(m+3)/2}=0$ and $Q_{(m+3)/2}=P_{(m+5)/2}=\frac{1}{2}(m+1)$.

On the other hand, if $Q_{(m+3)/2}=\frac{1}{2}(m+1)$, we have $P_{(m+3)/2}=\frac{1}{2}(m+3)$ and $B_{(m+3)/2}=0$. Recall the matching description of P in the proof of Lemma 8.3.

When $\lceil\frac{1}{2}(m+3)\rceil$ is reached all of $\{2, 3, \ldots, \lfloor\frac{1}{2}(m+3)\rfloor\}$ have been assigned except $\lfloor\frac{1}{2}(m+3)\rfloor$ and one lower value r. Let r be so defined for the remainder of this proof. Note that r is the only element less than $\lfloor\frac{1}{2}(m+3)\rfloor$ that can appear in P preceding $P_{\lceil(m+3)/2\rceil}$. In fact, it must begin the reversion containing $P_{\lceil(m+3)/2\rceil}$. In the case at hand, since we are matching $\frac{1}{2}(m+3)$ to itself, we also match r to itself. r cannot equal $\frac{1}{2}(m+1)$ since the later algorithm produced $Q_{(m+3)/2}=\frac{1}{2}(m+1)$. So $P_{(m+1)/2}>P_{(m+3)/2}$, and $P_{(m+1)/2}$ cannot begin a reversion. $P_{(m+3)/2}$ also does not (since $B_{(m+3)/2}=0$), so we have $S_{(m+1)/2}=0$ and $Q_{(m+1)/2}=P_{(m+3)/2}=\frac{1}{2}(m+3)$.

Q is now effectively complete. If $Q_{(m+1)/2}=\frac{1}{2}(m+1)$, there remains one unassigned position and element. Match it to itself. If $\frac{1}{2}(m+1)$ and $\frac{1}{2}(m+3)$ are matched to each other, two positions and the corresponding elements remain unassigned. Matching them to each other turns out to be the only legal way to complete Q. This is true because the larger of the unmatched elements is r, defined above. It did not appear in the latter half of P, hence also could not in Q. Since we already have $P_r=r$, we cannot also have $Q_r=r$.

Finally, we must show that the Q we have constructed satisfies Lemma 6.1. It already holds for all elements of Q beyond $\frac{1}{2}m$, and for r in the case $P_{\lceil(m+3)/2\rceil}=\lfloor\frac{1}{2}(m+3)\rfloor$. Suppose $Q_j=P_i=k$ with $i\leqslant j\leqslant\frac{1}{2}(m+1)$. Symmetry gives us $P_k\leqslant Q_k$, which is already false if $k>\frac{1}{2}(m+1)$. However, it can only happen once that $k\leqslant\frac{1}{2}(m+1)$ with $i\leqslant\frac{1}{2}(m+1)$: when $k=r$. Similarly, if $P_k\leqslant Q_k$ with $k\leqslant\frac{1}{2}(m+1)$ and $P_k>\frac{1}{2}(m+1)$, symmetry gives us a contradiction in the 'good' section. Again this can only happen once with $k\leqslant\frac{1}{2}(m+1)$ and $P_k\leqslant\frac{1}{2}(m+1)$: when $k=r$.

To summarize, we have shown that there are 2^q symmetric Q that can be paired with P that are realizable and determine symmetric quadruples. □

Theorem 8.5. *The number of quadruples realizable by symmetric NOHO-graphs on n vertices is* $3^{\lfloor(m-1)/2\rfloor}$.

Proof. If $B(P)$ has q ones, they may occur at any of the $t=\lfloor\frac{1}{2}(m-1)\rfloor$ steps in constructing P. The binomial theorem and the preceding lemma yield $\sum\binom{t}{q}2^q=3^t$ as the number of symmetric solutions. □

Knowing the number of symmetric quadruples, we can eliminate some of the overcounting of NOHO-graphs by quadruples. To determine the number of non-isomorphic NOHO-graphs, we must determine the other ways this can happen. For example, the last two graphs in Fig. 7.2 are isomorphic. Each quadruple is the reverse quadruple of the other. To eliminate this type of overcounting, we must count the 'reversible' quadruples. This will be done in Section 11 by examining $(S(G); T(G))$ instead of $(P(G); Q(G))$. For this we will need conditions for realizability of $(S; T)$, and for that we need to study 'irreducible' NOHO-graphs.

9. Concatenation of NOHO-graphs

We define a combining operator on NOHO-graphs.

Definition. Suppose G_1 and G_2 are NOHO-graphs on n_1 and n_2 vertices with canonical numberings $\{x_j^i\}$ and $\{y_j^i\}$. We define the *concatenation* of G_1 and G_2, denoted G_1+G_2, to be a new graph G_3 defined as follows.

To obtain G_3, begin with the union of G_1 and G_2 (i.e., unite the vertex sets and the edge sets of the two graphs). Delete the vertices x_0^2 and y_0^1 and the edges incident to them. Finally, identify two pairs of vertices: $l(x_0^2)$ with y_1^1 and $l(y_0^1)$ with x_1^2. Now G_3 is a graph on $n_3=n_1+n_2-4$ vertices, with $2n_1-4+2n_2-4-4=2n_3-4$ edges.

For the ordering of edges in G_3, let the first or last edges be the first or last edges that remain from G_1 or G_2. For the middle edges, preserve the order between any two edges that came from the same G_i. In addition, set every edge from $C^1(G_1)$ less than every edge from $C^1(G_2)$, and set every edge from $C^2(G_2)$ less than every edge from $C^2(G_1)$. Fig. 9.1 gives an example.

Note that concatenation is not a commutative operator. However, NOHO-graphs do form a non-commutative semi-group under concatenation. The 4-cycle is an identity element, associativity follows from the construction, and the next lemma verifies closure.

Lemma 9.1. *If G_1 and G_2 are NOHO-graphs, then the concatenation G_1+G_2 is also a NOHO-graph.*

Proof. We need only show that G_1+G_2 contains paths between all pairs of vertices and satisfies NOHO. We may consider the identified vertices as elements of either of the original graphs. Any path present in one of the component graphs is still present in G_1+G_2, unless it used one of the deleted vertices. the only paths that used them as non-endpoints are $(x_1^2, x_0^2, l(x_0^2))$ and $(y_1^1, y_0^1, l(y_0^1))$. In the concatenation these paths can be replaced by undeleted paths from the other summand, since $x_1^2=l(y_0^1)$ and $l(x_0^2)=y_1^1$.

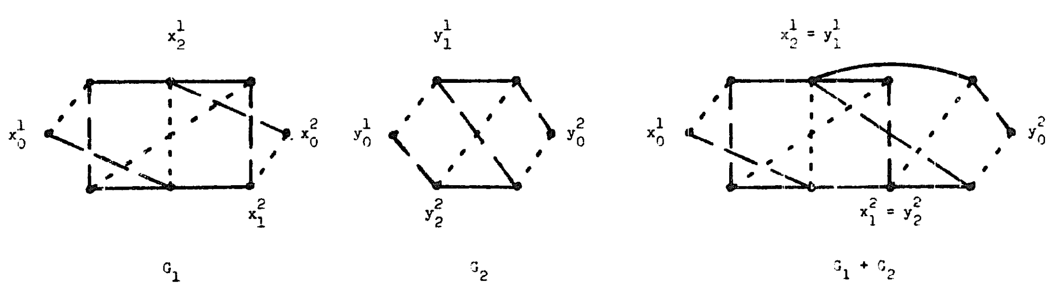


Fig. 9.1. Concatenation: (4231; 2301; 10; 01)+(321; 210; 1; 1)=(53421; 34210; 101; 011).

We easily obtain an increasing path from a vertex of G_i to a vertex in G_j. If v lies in G_1 and w in G_2, $v \to w$ can be formed by attaching $y_1^1 \to w$ from G_2 to the end of $v \to l(x_0^2)$ from G_1. Similarly, $w \to v$ can be obtained by linking the paths $w \to l(y_0^1)$ from G_2 and $x_1^2 \to v$ from G_1. These links produce increasing paths, because the only edges they use that are incident to the identified vertices were middle edges in the original graphs.

Finally, to verify NOHO we note that no increasing path that starts at a vertex from G_i can leave those vertices and later return. This would require travelling at least one edge along $C^i(G_1+G_2)$, crossing to $C^j(G_1+G_2)$, and returning. The crossover can only be a first or last edge, which prohibits including the later or earlier portion of the path. On the other hand, no path violating NOHO can lie entirely within the edges coming from one of the summands, since they are NOHO-graphs. □

Next we answer the natural question of how to determine the quadruple of sequences for the concatenated graph. It is clear from the construction that a vertex in G_1+G_2 lies on the distinguished path in $C^i(G_1+G_2)$ if and only if it did so in C^i in the appropriate summand. Hence we note the following:

Remark 9.2. *If NOHO-graphs G and G' are given by quadruples $(P; Q; S; T)$ and $(P'; Q'; S'; T')$, then*

(a) $S(G+G') = (S_2, \ldots, S_{m_1}, S'_2, \ldots, S'_{m_2})$,

(b) $T(G+G') = (T_2, \ldots, T_{m_1}, T'_2, \ldots, T'_{m_2})$.

In other words, concatenation concatenates the caterpillar sequences. It is to obtain this result that we ignore S_1 and T_1 and write T backwards. Note that the identified vertices lie on the distinguished paths in *both* summands, corresponding to the fact that the first and $m+1$st elements of S and T are always 1. As for $P(G+G')$ and $Q(G+G')$, they can be determined from S and T by the algorithm in Theorem 7.3. Alternatively, using this construction, they can be determined directly by dropping various elements from the component permutations, adding m_1-1 to the elements of P' and m_2-1 to those of Q, and concatenating.

The next lemma is an important companion to Remark 9.2, allowing us to 'deconcatenate' at will.

Lemma 9.3. *If $(S; T)$ determines a NOHO-graph G, then the initial segment $(S_2, \ldots, S_k; T_2, \ldots, T_k)$ determines a NOHO-graph if and only if the terminal segment $(S_{k+1}, \ldots, S_m; T_{k-1}, \ldots, T_m)$ also determines a NOHO-graph.*

Proof. We need only show one direction; the converse follows from reversing the edge ordering. The proof proceeds by examining two appropriate smaller graphs.

Define $c(k)$ conversely to $b(k)$, as the position of the first 1 in T *after* position

k. Let G_1 be the subgraph of G induced by

$$\{x_0^1, \ldots, x_k^1, l(x_{b(k+1)}^1), x_m^2, \ldots, x_{m+1-k}^2, x_{c(m+1-k)}^2\}.$$

Let G_2 be the subgraph of G induced by

$$\{f(x_{b(k+1)}^1), x_{b(k+1)}^1, x_{k+1}^1, \ldots, x_m^1, x_{m-k}^2, \ldots, x_0^2\}.$$

Let G_i inherit the edge ordering from G. Note that if $G = G_1 + G_2$, then $x_{b(k+1)}^1$ and $x_{c(m+1-k)}^2$ would be the vertices obtained by identifying pairs of vertices in the concatenation construction. Also $l(x_{b(k+1)}^1)$ and $f(x_{b(k+1)}^1)$ would have degree 2 in the induced subgraphs and would be the vertices deleted for concatenation. G_1 and G_2 are graphs composed of two caterpillars and matchings between them. The matchings consist of first and last edges but may not be complete. The caterpillar sequences are the initial and final segments mentioned in the statement of the lemma.

If the initial segment determines a NOHO-graph, G_1 is the graph it must be, since $(S; T)$ determines a *unique* NOHO-graph. Now consider G_2. It satisfies NOHO, since every path in G_2 appears also in G. To show it pools all the information, we show that an increasing path in G between vertices of G_2 uses only vertices of G_2. The only possible violator is a path between the caterpillars, say $x_r^2 \to x_s^1$ where neither vertex lies in G_1. As in the proof of Lemma 9.1, such a path starts by moving to $f(x_r^2)$ or ends by coming from $l(x_s^1)$, and the remainder of the path lies in the appropriate caterpillar. So, if the path enters G_1, we must have $f(x_r^2)$ or $l(x_s^1)$ in G_1. But G_1 is a NOHO-graph, so all the first and last neighbors of its vertices also lie in G_1. $\square$

Given this result, it is natural to call a realizable quadruple or NOHO-graph *irreducible* if it cannot be expressed as a concatenation of two smaller ones. In the next section we will enumerate subclasses of irreducible quadruples.

10. Irreducibility

Before counting irreducible quadruples, we introduce some standard terminology about compositions of integers. A *composition* of an integer p is an ordered sequence of positive integers, called its parts, which sum to p. In the application to NOHO-graphs, p will be related to n by $p = m - 1 = \frac{1}{2}(n-4)$, because $\frac{1}{2}(n-4)$ is additive under concatenation. The ith *partial sum* q_i of a composition is the sum of the first i parts. For example, the composition $(2, 4, 3, 5)$ has partial sums $(2, 6, 9, 14)$. A *refinement* of a composition of p is a composition of p with at least as many parts whose partial sums contain the partial sums of the original composition. $(2, 2, 2, 3, 3, 2)$ is a refinement of our original composition. The *least common refinement* of two compositions of p is the composition whose partial sums are the union of the partial sums of the original compositions. For example, the least common refinement of $(2, 3, 5)$ and $(1, 3, 1, 4, 1)$ is $(1, 1, 2, 1, 4, 1)$.

This terminology will be useful for the following lemma, which states a convenient fact about concatenation: NOHO-graphs are 'uniquely factorable' into irreducible pieces. In algebraic terms, this means the irreducible NOHO-graphs are the prime generators of the semigroup of NOHO-graphs under concatenation.

Lemma 10.1. *Any realizable quadruple can be uniquely expressed as a concatenation of irreducible quadruples.*

Proof. Any such decomposition of a quadruple breaks up (S, T) into segments which each determine NOHO-graphs. For example, describing NOHO-graphs as $G(S, T)$, we have

$$G(101010, 111101) = G(1, 1) + G(010, 111) + G(10, 01).$$

We can describe the decomposition by a composition of the integer $p = \frac{1}{2}(n-4)$.

Consider the smallest counterexample. Let r and s be the first partial sums of the two compositions. If $r = s$, then by Lemma 9.3 we can strip off the initial segment and have a smaller counterexample. If $r < s$, then $(S_2, \ldots, S_{r+1}; T_2, \ldots, T_{r+1})$ is a realizable initial segment of the realizable $(S_2, \ldots, S_{s+1}; T_2, \ldots, T_{s+1})$. By Lemma 9.3 again, the latter is not irreducible. The case $r > s$ is symmetric. □

Having proved unique decomposition, induction makes it easy to count various classes of solutions.

Theorem 10.2. *The number of realizable quadruples on n vertices formed by concatenating k irreducible quadruples is $\binom{p-1}{k-1}2^{p-k}$, where $p = \frac{1}{2}(n-4)$.*

Proof. By induction on p. For $p = 1$, $k = 1$, we have the one NOHO-graph on 6 vertices. Assume the theorem is true for smaller values than p.

First consider $k > 1$. To obtain such a quadruple we take a composition of p with k parts and fill the quadruple with irreducible $(S; T)$-segments of those lengths. By induction, each segment of length r can be filled by 2^{r-1} irreducible pairs. Filling each segment in all possible ways, Lemmas 9.1 and 10.1 say these are realizable, distinct, and exhaustive. So, for each composition $r_1 + \cdots + r_k = p$ with k parts, there are $2^{r_1-1} \cdots 2^{r_k-1} = 2^{p-k}$ quadruples of this type. There are $\binom{p-1}{k-1}$ compositions of p with k non-zero parts, so the total number of solutions is $\binom{p-1}{k-1}2^{p-k}$.

This holds also for $k = 1$. By the binomial theorem, 2^{p-1} of the 3^{p-1} quadruples counted in Theorem 7.5 remain. □

Let $\lfloor t \rfloor$ be the greatest integer $\leq t$ and $\lceil t \rceil$ the least integer $\geq t$.

Theorem 10.3. *The number of symmetric quadruples on n vertices formed by* *concatenating k irreducible parts is*

$$\binom{\lfloor \frac{1}{2}(p-1) \rfloor}{\lfloor \frac{1}{2}(k-1) \rfloor} 2^{\lfloor (p+1-k)/2 \rfloor},$$

except zero when $p=\frac{1}{2}n-2$ is odd and k is even. In particular, the number of symmetric irreducible quadruples is $2^{\lfloor p/2 \rfloor}$.

Proof. We use an induction similar to the proof above. For the basis, we have the quadruples on 6 and 8 vertices. Assume the theorem is true for smaller values than p.

First consider $k>1$. If k is even, p must be even to allow symmetry. We determine a composition of the first $\frac{1}{2}p$ places into $\frac{1}{2}k$ parts, fill it with irreducible $(S;T)$-segments of the appropriate lengths, and then obtain the rest by reflection. Again Lemmas 9.1 and 10.1 justify the conclusion that this counts everything. There are $\binom{p/2-1}{k/2-1}$ compositions and $2^{(p-k)/2}$ solutions for each one.

If k is odd and $k>1$, determine a composition of q with $r=\frac{1}{2}(k-1)$ parts, where $2q<p$. The middle segment of (S,T) will have length $p-2q$. In that segment we place a symmetric irreducible segment, of which by induction there are $2^{\lfloor (p-2q)/2 \rfloor}$. There are 2^{q-r} ways to fill the remainder. Again this yields the correct number. The total number of quadruples is

$$\sum_{q=1}^{\lfloor (p-1)/2 \rfloor} \binom{q-1}{r-1} 2^{q-r} 2^{\lfloor p/2 \rfloor - q} = 2^{(\lfloor p/2 \rfloor - r)} \sum_{p=1}^{\lfloor (p-1)/2 \rfloor} \binom{q-1}{r-1}$$
$$= \binom{\lfloor \frac{1}{2}(p-1) \rfloor}{r} 2^{\lfloor p/2 \rfloor - r}.$$

When $k=1$, we subtract the other values from $3^{\lfloor p/2 \rfloor}$, the total number of symmetric quadruples, to get the number of symmetric irreducible ones. □

11. Reversible NOHO-graphs

In this section we discuss the long-postponed conditions for realizability of the pair $(S;T)$. The condition for irreducible $(S;T)$ is easiest to express, and then the condition for $(S;T)$ in general can be obtained by concatenation. Afterwards, the characterization of irreducible $(S;T)$ enables us to count the quadruples which remain unchanged when the edge ordering is reversed.

We will see that when $m>2$, an irreducible $(S;T)$ begins and ends with $S_2 \neq T_2$ and $S_m \neq T_m$, but between those positions $S_k = T_k$. There is also a parity condition. As usual, we begin with a structural lemma. It characterizes the placement of edges in graphs determined by an $(S;T)$ which begins as described. Let the *weight* of a binary sequence be the number of 1's it contains.

Lemma 11.1. *Consider canonically numbered caterpillars determined by a pair $(S;T)$, though $(S;T)$ need not determine a NOHO-graph. Suppose $S_2 \neq T_2$, $S_r = T_r$ for $3 \leq r < k$, and paths of first and last edges are grown from x_0^1 by the algorithm in Theorem 7.3. Then for $2 \leq r < k$,*

(a) $S_r = 0 \Rightarrow x_r^1 \sim x_{m+2-r}^2$,

(b) $S_r = 1 \Rightarrow x_r^1 \sim x_{m+2-c(r)}^2$.

If $(S_2, \ldots, S_{r-1})$ has even weight, then the edge described is a first edge; otherwise, it is a last edge. Finally, each x_r^1 and x_{m+2-r}^2 with $r < k$ acquires two neighbors in the other caterpillar, and the subscript of the vertex assigned as $f(x_r^1)$ is at most $m+3-r$.

Proof. By induction on r. The basis requires verifying the result through the first 1 in $(S_2, \ldots, S_m)$, if there is one. This follows from Corollary 6.2, from the argument for strings of 0's two paragraphs below, and from $l(x_1^1) = x_{m+2-c(1)}^2$. We omit the details. The induction will assume the result is true for $b(r), \ldots, r$ and prove it for $r+1, \ldots, \min\{c(r), k-1\}$. We may assume $S_r = 1$. The proof proceeds by growing the alternating path from $x_{b(r)}^1$ along the edge guaranteed by the induction hypothesis, using the algorithm in Theorem 7.3. That algorithm is defined so that the results satisfy Lemma 7.1 and the application of Lemma 7.1 to the reflected graph, which we now call Lemma 7.1*. Lemma 7.1* states that

(a) $T_j = 0 \Leftrightarrow f(x_{m+2-j+1}^2) = l(x_{m+2-j}^2)$,

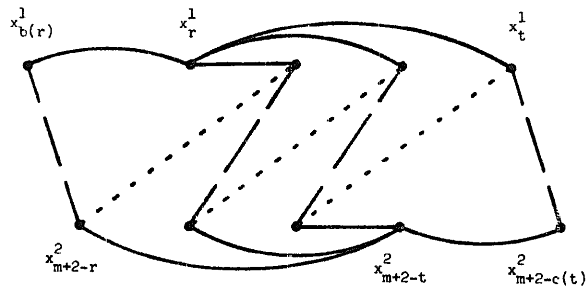
(b) $T_j = 1 \Leftrightarrow f(x_{m+2-j+1}^2) = l(x_{m+2-c(j)}^2)$.

The appropriate figure to illustrate this is obtained by rotating Fig. 7.1 by 180°. Recall that $T_j = 1$ if and only if x_{m+2-j}^2 lies on the distinguished path in $\bar{C}^2$. The discussion of how the path grows from $x_{b(r)}^1$ depends heavily on Lemmas 7.1 and 7.1*. Every time we grow the path to a vertex in C^1 we use Lemma 7.1, and every time it grows to C^2 we use Lemma 7.1*. The two cases below are illustrated in Figure 11.1.

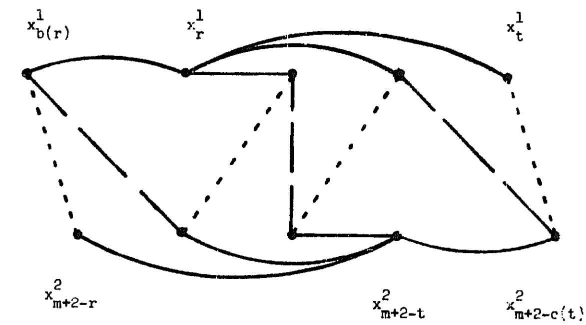
Suppose first that $(S_2, \ldots, S_{r-1})$ has non-zero even weight. Then $(S_2, \ldots, S_{b(r)-1})$ has odd weight, and by induction $(x_{b(r)}^1, x_{m+2-r}^2) \in L(G)$. Next $(x_{m+2-r}^2, x_{r+1}^1) \in F(G)$, since $S_r = 1$. If $T_{r+1} = 0$, we also get $(x_{r+1}^1, x_{m+2-(r+1)}^2) \in L(G)$, as desired. Continuing, if $S_{r+1} = T_{r+2} = 0$ we have $(x_{m+2-(r+1)}^2, x_{r+2}^1) \in F(G)$ and $(x_{r+2}^1, x_{m+2-(r+2)}^2) \in L(G)$, and so on until the next 1. Let $t = c(r)$ be the next 1, if it is less than k, so $S_t = T_t = 1$. If $t > r+1$, $S_{t-1} = 0$ gives $(x_{m+2-t+1}^2, x_t^1) \in F(G)$, while if $t = r+1$ the same fact was given above by $S_r = 1$. Now $T_t = 1$ gives $(x_t^1, x_{m+2-c(t)}^2) \in L(G)$ as desired.

On the other hand, if $(S_2, \ldots, S_{r-1})$ has odd weight, then $(S_2, \ldots, S_{b(r)-1})$ has even weight, and by induction $(x_{b(r)}^1, x_{m+2-r}^2) \in F(G)$. If $T_{r+1} = 0$, we get $(x_{b(r)}^1, x_{m+2-(r-1)}^2) \in L(G)$. Then $S_r = 1$ gives $(x_{r+1}^1, x_{m+2-(r+1)}^2) \in F(G)$, the desired result for $S_{r+1} = 0$. Again let $t = c(r)$. Continuing as in the previous case, $T_{r+2} = S_{r+1} = 0$ gives $(x_{r+1}^1, x_{m+2-(r+1)}^2) \in L(G)$ and $(x_{m+2-(r+2)}^2, x_{r+2}^1) \in F(G)$ and so on until $T_t = S_t = 1$. If $t > r+2$, then $S_{t-2} = 0$ gives $(x_{t-1}^1, x_{m+2-(t-1)}^2) \in F(G)$. If $t = r+2$, the same fact was given above by $S_r = 1$. Now, for $t \geq r+2$, $T_t = 1$ gives $(x_{t-1}^1, x_{m+2-c(t)}^2) \in L(G)$. On the other hand, if $t = r+1$, then $T_t = 1$ gives

(a)



(b)



(c)

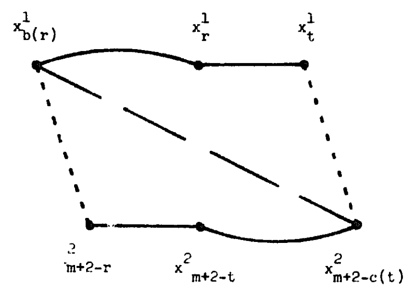


Fig. 11.1. $(S_2, \ldots, S_{r-1})$ has even or odd weight.

$(x^1_{b(t-1)}, x^2_{m+2-c(t)}) \in L(G)$, as in the third graph in Fig. 11.1. Finally, having reached $x^2_{m+2-c(t)}$ along a last edge from the appropriate vertex, we can apply $S_{t-1} = 0$ when $t > r+1$ or $S_{t-1} = 1$ when $t = r+1$ to get $(x^1_t, x^2_{m+2-c(t)}) \in F(G)$, as desired.

Finally, the easiest way to see that two edges are generated for each vertex is to look at Fig. 11.1. All the cases where S_r or T_r equals 0 or 1 and the weight of $(S_2, \ldots, S_{r-1})$ is odd or even are covered. The same is true for the claim that $f(x^1_r) = x^2_s$ for some $s \leq m+3-r$. $\square$

Lemmas 11.5 and 11.6 are the main results on realizability of these sequences. First we must make some minor comments.

Remark 11.2. *In a realizable quadruple, $P_i = m+2-i$ for all $1 \leq i < r$, where S_r is the first* 1 *in S.*

Proof. $P_1 = m+2-1$, so the first time P_i is less than $m+2-i$ begins a reversion of length at least two. This forces $S_i = 1$ by the condition of Lemma 6.7 on $(P;S)$. □

Corollary 11.3. *In a realizable quadruple,* $(S_2; T_2) \neq (0;0) \neq (S_m; T_m)$.

Proof. If $S_2 = 0$, Remark 11.2 says $f(x_2^1) = x_m^2$. But $f(x_2^1) = l(x_0^1)$ by Corollary 6.2, so $T_2 = 1$. $S_m = 0$ implies $T_m = 1$ by applying the same argument to the reverse edge ordering. □

Corollary 11.4. *In a realizable quadruple,* $S = (S_2, \ldots, S_m)$ *cannot be identically zero. Also, there cannot be two* 1*'s in* T *before the first* 1 *in* S *or after the last* 1 *in* S. *The same results hold with* S *and* T *interchanged.*

Proof. By reversing and reflecting and by Corollary 11.3, we need only show there cannot be two ones in T before a 1 is encountered in S. By Corollary 11.3 and Remark 11.2 these would be T_2 and T_i with $S_i = 0$ and $f(x_i^1) = x_{m+2-i}^2$. Since $T_2 = 1$, applying Lemma 7.1* gives $l(x_1^1) = x_{m+2-c(2)}^2 = x_{m+2-i}^2$. This creates a triangle on $(x_1^1, x_i^1, x_{m+2-i}^2)$, as shown in Fig. 11.2.

Lemma 11.5. *Suppose* $\{S_2, T_2\} = \{S_k, T_k\} = \{0, 1\}$ *and* $S_r = T_r$ *for* $2 < r < k$. *Then* $(S_2, \ldots, S_k; T_2, \ldots, T_k)$ *is not a realizable initial segment of a realizable quadruple if this segment of* S *(or of* T*) has even weight.*

Proof. Suppose it is a realizable initial segment and has even weight. By Corollary 11.4, these initial segments each have at least two 1's. We have two cases, which are illustrated in Fig. 11.3.

Suppose first that $(S_k; T_k) = (0; 1)$, so applying Lemma 11.1 gives $(x_{b(k)}^1, x_{m+2-k}^2) \in L(G)$. Since the last 1 in S marks the index of $l(x_0^2)$, we must have another 1 in S after $b(k)$; let it be S_t. Note that $t > k$. Since $b(t) = b(k)$, applying Lemma 7.1 yields $(x_{m+2-k}^2, x_{t+1}^1) \in F(G)$, i.e. $P_{t+1} = m+2-k$. Now, if $i < t$, then $P_i > m+2-k$. Otherwise, P_i, P_{t+1} would form an increasing pair. By Lemma 6.3,

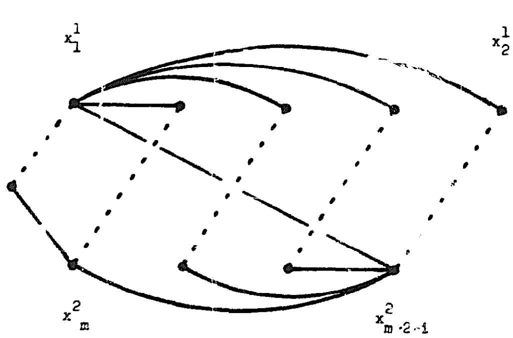


Fig. 11.2. Too many ones in $T(G)$.

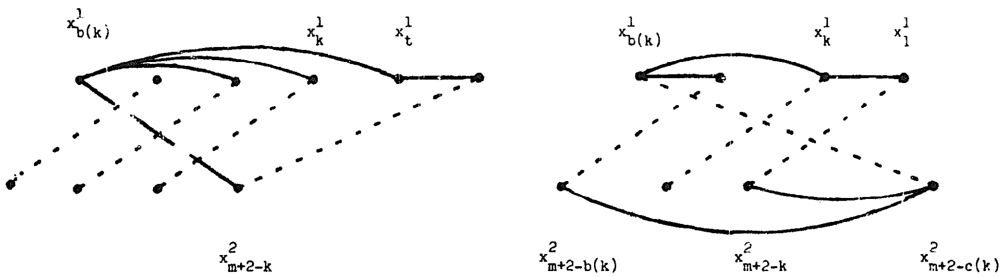


Fig. 11.3. Even weight $(S; T)$ with $(S_k; T_k) = (0; 1)$ or $(1; 0)$.

this would require $x_i^1 \sim x_{t+1}^1$, but when $S_t = 1$ the only neighbor of x_{t+1}^1 preceding it in C^1 is x_t^1. With $t > k$ there are at least k such preceding positions in P, but only $k-1$ values exceeding $m+2-k$ available for them.

On the other hand, suppose $(S_k; T_k) = (1; 0)$. In this case Lemma 11.1 requires $(x_{b(k)}^1, x_{m+2-c(k)}^2) \in F(G)$. Equivalently, $P_{b(k)} = m+2-c(k)$, and hence $m+2-c(k) < m+3-b(k)$. By Remark 8.2, this means $P_{b(k)}$ begins a reversion, and $P_i = m+3-i$ for $b(k) < i \leq c(k)$. Since $c(k) > k$, the reversion extends past P_k. P_k neither begins nor ends the reversion, so by Lemma 6.4 S_k could not be 1. □

Lemma 11.6. *Suppose* $\{S_2, T_2\} = \{S_m, T_m\} = \{0, 1\}$ *and* $S_q = T_q$ *for* $2 < q < m$. *If* $(S_2, \ldots, S_m)$ *has odd weight, then* $(S; T)$ *generates an irreducible quadruple.*

Proof. We first note that if the segment is realizable, it must be irreducible. If not, consider a possible decomposition. $(0; 0)$ cannot begin or end a realizable sequence, and by Lemma 9.3 a realizable $(1; 1)$ can be removed from the begining or end of a realizable $(S; T)$ to leave a realizable $(S'; T')$. So, this segment will be reducible if and only if the interior consists entirely of 1's and the exterior $(0; 1)$ or $(1; 0)$ are realizable. But $(1; 1)$ is the *only* realizable $(S; T)$ with $m = 2$.

Note also that the problems with even numbers of 1's don't arise. With an odd number the alternating paths end nicely, as shown in Fig. 11.4.

As noted in Lemma 11.1, constructing first edges and last edges by the algorithm in Theorem 7.3 generates complete matchings: a first neighbor and a

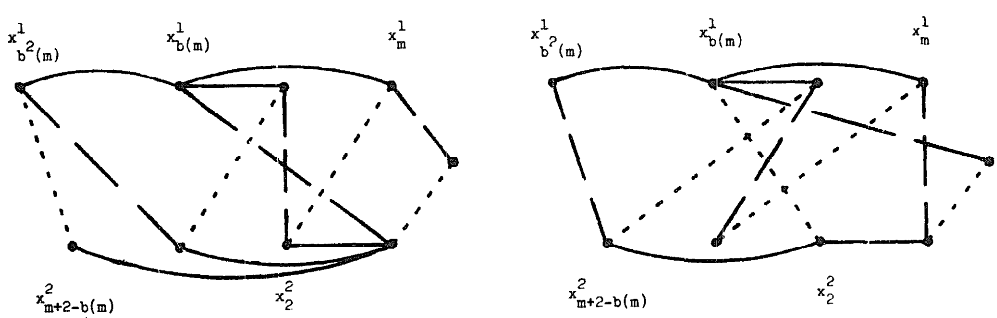


Fig. 11.4. Examples of $(S; T)$ with odd weight, ending $(1; 0)$ or $(0; 1)$.

last neighbor for each vertex. Therefore, P and Q exist, especially with the noncontradictory ending shown in Fig. 11.4. The output of this algorithm always satisfies NOHO. In fact, it also satisfies Lemmas 6.1, 7.1, and 7.1*.

Rather than proving realizability by tediously verifying the existence of all increasing paths, this time we show that the remaining conditions for sufficiency hold. If the graphs that are produced satisfy Lemmas 6.5 and 6.7, then they also satisfy Remark 6.10, since the reflected graph also has the form described. By Theorem 7.4, this suffices to show that the resulting quadruple is realizable.

To prove the reversions are all simple, suppose $(P_i, P_j, P_k) = (r, s, t)$ with $i<j<k$, $r<t$, $s<t$. On each alternating path grown from x_0^1, the indices in C^1 strictly increase and in C^2 strictly decrease. This means neither path crosses itself when drawn in the usual fashion, so (x_k^1, x_t^2) is on one path, and (x_i^1, x_r^2) and (x_j^1, x_s^2) are on the other, with $r>s$. Looking at such a pair of edges which are closest together on the path, we may assume the path proceeds from one to the other via $(x_i^1, x_s^2) \in L(G)$. (See Fig. 11.5.)

Again we have two cases, but in both we contradict the requirement of Remark 8.2 that $k+P_k \leq m+3$. If $S_i = 0$, then Lemma 11.1 implies that $x_i^1 \sim x_{m+2-i}^2$, so r or s is $m+2-i$. But $k \geq i+2$ and $t \geq \max\{r, s\}+1 \geq m+3-i$, so $k+P_k \geq m+5$. On the other hand, suppose $S_i = 1$. Since $P_i = Q_i$, S_{j-1} must also equal 1, and in fact $b(j-1) = i$. Since $S_q = T_q$ for $2<q<m$, we have $j-1 = c(i)$. Now we apply Lemma 11.1 again. Since $x_i^1 \sim x_{m+2-c(i)}^2$, r or s equals $m+2-(j-1)$. This time $k \geq j+1$, so again $k+P_k \geq j+1+m+3-(j-1) = m+5$.

Now it is easy to verify Lemma 6.7. If all the reversions are simple, a reversion of length more than one can begin only when $P_i < m+3-i$. In the proof of Lemma 11.1 this could happen only when $S_i = 1$ (in fact, an odd 1), as desired. Until the next 1, $P_i = m+2-i+1$ in the construction. Thus, before the end of the reversion, S_i must be 0, as desired. □

To summarize, we have proved

Theorem 11.7. *(S, T) determines an irreducible NOHO-graph on $2m+2$ vertices if and only if $\{S_2, T_2\} = \{S_m, T_m\} = \{0, 1\}$, $S_k = T_k$ for $3 \leq k < m$, and S and T have odd weight.*

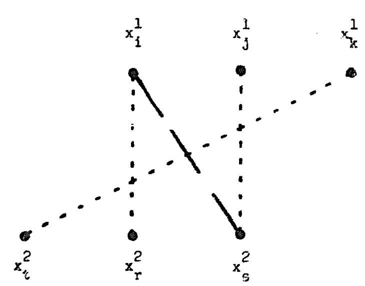


Fig. 11.5. A potential bad triple.

Corollary 11.8. *There are 3^{m-2} realizable quadruples, 2^{m-2} of which are irreducible.*

Proof. Theorem 11.7 provides direct proofs of Theorems 10.2 and 7.5, which counted the number of quadruples concatenated from k irreducible quadruples and the number of irreducible quadruples. We no longer need the binomial theorem. To generate an irreducible quadruple, the segment $(S_3, \ldots, S_{m-1})$ can be chosen arbitrarily, in 2^{m-3} ways. There are then two ways to fill in S_2, T_2, S_m, T_m to have odd weight in each sequence.

To generate a realizable quadruple, we grow $(S; T)$ from $(S_2; T_2)$ with three choices at each step from 2 through $m-1$. When $k=2$ or $(S_{k-1}; T_{k-1})$ ends an irreducible component, we can choose $(0; 1)$, $(1; 0)$, or $(1; 1)$ for $(S_k; T_k)$. When $(S_{k-1}; T_{k-1})$ does not end a component, we can choose $(0; 0)$, $(1; 1)$, or the proper one of $(0; 1)$ and $(1; 0)$ to end the component with odd weight. When $(S_m; T_m)$ is reached there is only one choice for it, since the last component must end there. □

Recall that reversible quadruples are those for which $S_k = S_{m+2-k}$ and $T_k = T_{m+2-k}$. An argument very closely paralleling that made for Theorem 10.3 gives us the following theorem, but this time the irreducible case can be done directly.

Theorem 11.9. *The number of reversible quadruples formed by concatenating k irreducible parts is*

$$\binom{\lfloor \frac{1}{2}(p-1) \rfloor}{\lfloor \frac{1}{2}(k-1) \rfloor} 2^{(p-k)/2}$$

if the parity of p and k is the same, otherwise it is zero. In particular, the number of irreducible reversible quadruples is $2^{(p-1)/2}$ for odd p.

The total number of these is of greater interest.

Corollary 11.10. *The number of reversible quadruples on n vertices is $3^{\lfloor (p-1)/2 \rfloor}$, where $p = \frac{1}{2}n - 2$.*

Proof. This follows from Theorem 11.9 by the binomial theorem, or directly by the same three-choice argument given in Corollary 11.8, stopping when the middle is reached. □

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[1] D.B. West, On a class of solutions to the gossip problem I, Discrete Math. 39 (1982) 307–326.